NAVAL POSTGRADUATE SCHOOL

MONTEREY, CALIFORNIA

THESIS

A GAME THEORETIC APPROACH TO CONVOY ROUTING

by

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June 2009

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REPORT DOCUMENTATION PAGE	*Form Approved OMB No. 0704-0188*

1. AGENCY USE ONLY *(Leave blank)*	**2. REPORT DATE** June 2009	**3. REPORT TYPE AND DATES COVERED** Master's Thesis

4. TITLE AND SUBTITLE A Game Theoretic Approach to Convoy Routing

5. FUNDING NUMBERS

6. AUTHOR(S) Smith, William T.

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Naval Postgraduate School
Monterey, CA 93943-5000

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING /MONITORING AGENCY NAME(S) AND ADDRESS(ES)
N/A

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES The views expressed in this thesis are those of the author and do not reflect the official policy or position of the Department of Defense or the U.S. Government.

12a. DISTRIBUTION / AVAILABILITY STATEMENT
Approved for public release; distribution is unlimited

12b. DISTRIBUTION CODE
A

13. ABSTRACT (maximum 200 words)

A two-person search / ambush game is considered where each player wants to maximize his survival time while minimizing the survival time of his adversary. This is done in the context of convoy routing where each player can choose which route they take. Their estimated survival times depend upon (a) if their adversary is directly searching on that route, (b) the indirect probability of detection or hazard if their adversary is not along that route, and (c) the risk involved with moving from route to route. It is possible for a player to be interdicted even if his adversary is not on that route. Each player has a payoff matrix that maximizes their expected time to capture. We show that both payoff matrices can be evaluated as a bimatrix game that yields optimal mixed Nash Equilibria through the use of non-linear programming. The results of this evaluation can be used to optimally conduct route clearing and convoy routing.

14. SUBJECT TERMS

Game Theory, Bimatrix, modeling, convoy routing, Nash Equilibrium, search, ambush

15. NUMBER OF PAGES
59

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	**18. SECURITY CLASSIFICATION OF THIS PAGE**	**19. SECURITY CLASSIFICATION OF ABSTRACT**	**20. LIMITATION OF ABSTRACT**
Unclassified	Unclassified	Unclassified	UU

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std. 239-18

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A GAME THEORETIC APPROACH TO CONVOY ROUTING

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Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN APPLIED MATHEMATICS

from the

NAVAL POSTGRADUATE SCHOOL
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A two-person search/ambush game is considered where each player wants to maximize his survival time while minimizing the survival time of his adversary. This is done in the context of convoy routing where each player can choose which route they take. Their estimated survival times depend upon (a) if their adversary is directly searching on that route, (b) the indirect probability of detection or hazard if their adversary is not along that route, and (c) the risk involved with moving from route to route. It is possible for a player to be interdicted even if his adversary is not on that route. Each player has a payoff matrix that maximizes their expected time to capture. We show that both payoff matrices can be evaluated as a bimatrix game that yields optimal mixed Nash Equilibria through the use of non-linear programming. The results of this evaluation can be used to optimally conduct route clearing and convoy routing.

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LIST OF VARIABLES

Search Model – Red's Expected Survival Time Without Blue On The Route

t..Time

$A_i = A_i(t, \alpha_i, q_i)$..............Expected survival time for Red along route i when Blue is not on the route

α_i.......................................Linear rate of increase for probability of indirect detection along route i

q_i.......................................Initial probability of indirect detection for Red along route i

$g_i(t) = q_i + \alpha_i \cdot t$...................Function governing the increase of risk along route i with time when Blue is not on the route

T_i.......................................Departure time for Red if he is on route i without Blue

U.......................................Limit of integration for evaluating the accumulated time Red stays on a route without Blue

Search Model – Red's Expected Survival Time With Blue on the Route

$B_i = B_i(t, \beta, p_i)$....................Expected survival time for Red along route i when Blue is on the same route

β..Linea rate of increase for probability of direct detection along route i

p_i.......................................Initial probability of direct detection along route i

$f_i(t) = p_i + \beta \cdot t$.....................Function governing the increase of risk along route i with time if Blue is on the route

D_i.......................................Departure time for Red if he is on route i with Blue

W.......................................Limit of integration for evaluating the accumulated time Red stays on a route with Blue

V_i......................................Value assigned to a route taking into account all other routes with greater expected survival times

Ambush Model – Expected Survival Time For Blue Without Red On The Route

$C_i = C_i\left(t, \gamma, r_i\right)$.....................Expected survival time for Blue along route *i* when Red is on the same route

γ.......................................Linear rate of increase for probability of hazard along route *i*

r_i......................................Initial probability of indirect hazard for Blue along route *i*

$h(t) = r_i + \gamma \cdot t$.....................Function governing the increase of risk along route *i* with time if Red is not on the route

M_i......................................Departure time for Blue from route *i* if Red is not on the route

K.......................................Limit of integration for evaluating the accumulated time Blue stays on a route without Red

Ambush Model – Expected Survival Time For Blue With Red On The Route

$E_i = E_i\left(t, \gamma, j_i\right)$.....................Expected survival time for Blue along route *i* with Red on the route

$j_i(t) = s_i + \gamma t$ ....Function governing the increase of risk along route *i* with time if Red is on the route

s_i ....Initial probability of direct hazard along route *i*

N_i......................................Time of departure for Blue from route *i* if Red is on the route

L......................................Limit of integration for evaluating the accumulated time Blue stays on a route with Red

Bimatrix Model

Ψ where $\psi_{ij} = \begin{cases} A_i & i \neq j \\ B_i & i = j \end{cases}$ ....The Search Model's Payoff Matrix

Λ where $\lambda_{ij} = \begin{cases} C_i & i \neq j \\ E_i & i = j \end{cases}$ ....The Ambush Model's Payoff Matrix

x^* ....Optimal mixed strategy for Red

y^* ....Optimal mixed strategy for Blue

$V_{Red} = x^{*T} \Psi y^*$ ....Value of the game for Red

$V_{Blue} = x^{*T} \Lambda^T y^*$ ....Value of the game for Blue

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I. INTRODUCTION

A. BACKGROUND

Military convoys, also referred to as Combat Logistic Patrols (CLPs), have always been lucrative targets for adversaries that do not have the military advantage to sustain and win in direct conflict. The safety of a convoy is often seen as a function of active measures employed by the convoy such as speed, increased aggressiveness and patrolling. To this end, Matthew Hakola used agent based modeling and principal component analysis to investigate which variables carry more weight in the success of the convoy [1]. However, when faced with multiple routes a commander must make decisions based on the threat associated with each route before choosing which one to take. William Ruckle's geometric approach to the game theory behind a hunter and prey has been useful in understanding the foundation of our game [2].

Our convoy, which we will name Blue, wishes to traverse an area but faces the threat of being intercepted between the start and finish of his route by an enemy lying in wait. We will call the enemy Red. Using Ruckle's example, we will assume our area is a unit square, with a horizontal x-axis and vertical y-axis. We will assume that neither Red nor Blue receive any intelligence on the position of the other once Blue starts to move. Because of this limitation Red gains no advantage from changing locations while Blue is in motion and so we will consider Red's position fixed. Blue simply wishes to get from $x = 0$ to $x = 1$. With this in mind we can disregard Red's x-coordinate, as it does not affect his ability to intercept Blue. Instead, we focus on Red's ability to intercept Blue as a function of his y-coordinate and range of his weapon, which we will define as r. Therefore, our unit box can now be decomposed into routes of $1/r$ width. Blue gains no advantage by changing routes while traversing through the area but can change routes with each new crossing. Using this logic, we limit our game to the use of individual routes versus a network of routes.

The limitation with most ambush models is that the threat is always one sided. This is evident in Ruckle's examples [2], as well as in textbook examples on game theory [6]. In reality, an ambusher must also contend with the fact that he may be discovered before he gets the opportunity to conduct an ambush. Furthermore, the threat is not always from an active searcher (also referred to as direct detection). Rewards and humanitarian assistance can influence an area to be more pro-active in uncovering and turning in cells that are planning ambushes (also referred to as indirect detection). So, the game becomes more complex with each side trying to maximize their own survival time while minimizing their opponent's time considering both indirect and direct detection.

We approach this model using a "deductive" search methodology following work done by Owen and McCormick [3]. This search methodology focuses on determining those routes most favorable to each player rather than trying to follow any "trail" left by their presence. Their algorithm provides us with an expected time to capture / ambush for both players. This time is dependent on the probability of both direct and indirect detection as well as the ability to successfully change routes. Then, given the expected time to capture / ambush for each player along each route, we use non-linear programming to determine the optimal strategy for each player to adopt in order to maximize individual survival time.

B. RESEARCH GOAL

The goal of this paper is to provide a way to investigate and determine the mixed strategies used by both the convoys and adversaries along a relatively small number of routes. This will allow commanders to make informed decisions on which routes to take to maximize the survival rate of CLPs as well as which routes are most likely to be ambush sites.

C. BASIC OVERALL MODEL

The approach we will use is that of a bimatrix game with the goal of finding equilibrium points. Our players are defined as Blue, for the convoy, and Red, for the ambusher. Each player has their own payoff matrix where their goal is to maximize their survival time while their adversary simultaneously tries to minimize it. We define $\boldsymbol{\Psi}$ as the payoff matrix for the Search Model where Red wishes to maximize his survival time while Blue searches for him. Our other payoff matrix, $\mathbf{\Lambda}$, represents the Ambush Model where Blue wishes to maximize his survival time while Red attempts to ambush him. These two payoff matrices are then combined into a bi-matrix model ($\mathbf{\Psi}$, $\mathbf{\Lambda}$) that represents the competing goals for each player (i.e., maximize individual survival time while minimizing the opponents).

The Search and Ambush payoff matrices are constructed by adopting a model developed by Owen and McCormick [3]. Their manhunting model considers a fugitive who faces not only the threat of being captured directly but also the threat of being turned in by the local populace [3]. The Search Model adapts this directly for Red's threat of being found through Blue's direct search and Blue's efforts to uncover him indirectly (i.e., through the actions of others). The Ambush Model takes into account the threat Blue faces from a direct ambush by Red as well as the indirect hazards that may prevent a convoy from being completed (terrain, length of route, etc.). Each route presents four initial probabilities:

1) The probability Red is detected indirectly by a third party (indirect detection (q))
2) The probability Red is detected directly by Blue (direct detection (p))
3) The probability Blue fails complete the convoy because of reasons other than an ambush (indirect hazard (r))
4) The probability Blue fails to complete the convoy because he was ambushed by Red (direct hazard (s))

The rate by which each initial probability increases is dependent upon the intensity of the efforts of the adversary. The intensity of each player's efforts to minimize his opponent's survival time is represented by the variables $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$. The amount of money and resources Blue spends on gaining the assistance of the local populace to uncover Red affects how quickly the threat of indirect detection increases. This rate of increase is represented by $\boldsymbol{\alpha}$ and can be seen as quantifying the aggressiveness of Blue's efforts to indirectly detect Red. The intensity of a direct search for Red (possibly as a function of the numbers of Soldiers, sensors, etc. involved) determines how quickly the threat of direct detection increases along a certain route and is represented by **β,** or the aggressiveness of Blue's efforts to directly uncover Red. We assume that the quality of the route (which affects the indirect threat to the convoy) will not change with repeated iterations along that route and therefore we limit ourselves to only one parameter for threat's rate of increase in the Ambush model. With every use of a route, Blue faces the risk of Red moving onto that route to intercept him on the next convoy. The rate at which this threat increases is defined as **γ—**Red's aggressiveness.

Once we construct the individual payoff matrices, we construct the bimatrix model (**Ψ**, **Λ**) where each cell contains the pair of values from the respective individual payoff matrices. Using non-linear programming, we determine the optimal route selection strategies for each player. We show that these optimal strategies (also known as Nash Equilibria) are dependent upon the desired survival time for each player, which can be viewed as each player's decision to be risk averse or risk prone. These optimal strategies are then used to determine which routes a convoy should take, as well as which routes a patrol can most expect to uncover the enemy.

II. SEARCH MODEL

A. OVERVIEW

We use a model created by Owen and McCormick [3] to determine the expected time until capture (or ambush) for each player. For the benefit of the reader, and to add our own notation that will be useful later in analysis, we will cover it again in Chapters II and III of this thesis. For our purposes, we define "indirect detection" as the threat Red faces of being discovered indirectly and "direct detection" as the threat he faces from being found directly by Blue.

1. Indirect Detection

The local populace of a region can pose a risk to Red and shorten his survival time. Blue can take advantage of this in a number of ways. He could offer greater rewards for revealing Red or he could gain the confidence of the local population so that it is in their best interest to betray Red. We can safely assume that as Red occupies an area the risk of discovery increases. Assuming that Blue is looking on another route for Red, Red's probability of detection and capture within *t* units of time on route *i* will be represented by $Q_i(t)$ and the probability he has not already been captured is $1-Q_i(t)$. This probability of capture is not static but rather increases with time. We let $g_i(t)$ represent the rate at which this probability increases along route *i* while Red is on it. Making the additional assumption that Red's risk is initially zero when he enters the route we get the following differential equation and its solution.

$$\begin{aligned} &Q_i'(t) = g_i(t)(1-Q_i(t)),\\ &Q_i(0) = 0 \\ \\ &Q(t) = 1-\exp\{-G(t)\} \end{aligned} \tag{1}$$

Note that the derivative of $Q_i(t)$ is the increase in the probability of detection. It is the probability that Red has not been detected multiplied by the rate of increase for that route. We further define g_i as a linear, strictly increasing, unbounded function with the initial probability, q_i, that Red will be captured on that route plus some linear rate of increase, $\alpha_i \geq 0$, that controls how quickly that probability increases with time. Owen and McCormick used a constant $\alpha = 0.01$ in their examples as a reasonable rate of increase. In the case of our game, we can assign a value slightly larger if Blue is very aggressively pursuing indirect means of detection along that route. Clearly, the value can change for each route based on Blue's efforts. G_i is the anti-derivative of g_i evaluated from zero to time *t*.

This gives us

$$\begin{aligned} g_i(t) &= q_i + \alpha_i t \\ G_i(t) &= \int_0^t g_i(s)ds = \mathrm{t}q_i + \frac{\alpha_i t^2}{2} \end{aligned} \tag{2}$$

Applying (2) to the solution in (1) we obtain

$$\begin{aligned} Q_i(t) &= 1 - \exp\{-G_i(t)\} \\ Q_i\,'(t) &= g_i(t)(1 - Q_i(t)) = g_i(t)\exp\{-G_i(t)\} \end{aligned} \tag{3}$$

Red wants to maximize his survival time, so we can assume that at some time T he will decide to move, if he has not already been captured. Keep in mind that Q(T) is the probability Red is captured by time T, 1-Q(T) is the probability that Red will be able to move at time T (i.e., he is not captured). We will assign the random variable X to represent the time Red spends in route i before moving. In determining this time, we must take into account not only the probability he will be able to move at that time, but also the probability he has not been captured before time T given the density $Q_i^{'}(t)$ from 0 to T. This gives us the formula for our expected time Red spends on route i.

$$E[X_i] = \int_0^T tQ_i^{'}(t)dt + (1 - Q_i(T))T \tag{4}$$

We can simplify this even further using (3)

$$E[X_i] = \int_0^T tg_i(t)\exp\{-G_i(t)\}dt + T\exp\{-G_i(T)\} \tag{5}$$

Red's move does not come without risk. He can always be detected en route to his next ambush location. Independent of when he moves, he expects to survive an additional V_i units of time after starting the move. Given that the probability that he will even get to move is $\exp\{-G_i(t)\}$, we can define his total expected survival time on route *i* as

$$A_i = \int_0^T t g_i(t) \exp\{-G_i(t)\} dt + T \exp\{-G_i(T)\} + \exp\{-G_i(T)\} V_i \tag{6}$$

Clearly Red's decision to depart is based upon his desire to maximize the time *T* of his departure. To maximize this we differentiate A_i with respect to *T*.

$$\frac{dA_i}{dT} = T g_i(T) \exp\{-G_i(T)\} + \exp\{-G_i(T)\} - T g_i(T) \exp\{-G_i(t)\} - g_i(T) \exp\{-G_i(T)\} V_i$$

When simplified we get

$$\frac{dA_i}{dT} = \exp\{-G_i(T)\} - g_i(T) \exp\{-G_i(t)\} V_i \tag{7}$$

Setting this to zero and solving for *T* produces the following

$$T_i = g_i^{-1}\left(\frac{1}{V_i}\right) \tag{8}$$

Since g_i is a strictly increasing function, we can be assured that T_i is unique. Now we notice that we can simplify equation (6) if we integrate by parts letting $u = t$ and $dv = g(t)*\exp\{-G(t)\}$.

Note: $\int_0^T t g_i(t) \exp\{-G_i(t)\} dt = -T \exp\{-G(T)\} + \int_0^T \exp\{-G_i(t)\} dt$

So Red's expected time to indirect detection along route *i* becomes

$$A_i = \int_0^T \exp\{-G_i(t)\} dt + \exp\{-G_i(T)\} V_i \tag{9}$$

Keeping in mind how we defined g_i we can take equation (8) and rewrite it as

$$q_i + \alpha_i T_i = \frac{1}{V_i}$$

Solving for T we get

$$T_i = \frac{1 - q_i V_i}{\alpha_i V_i} \tag{10}$$

We can see that T_i has the potential of being a negative number or zero. Since we assume that α_i, V_i are positive, this can only occur when the initial probability q_i is sufficiently large so $V_i \le q_i^{-1}$. Such a cell would present a significant risk to Red, and offer no gain in his expected survival time. Red would immediately move, if he found himself on such a route. We will therefore choose T_i by (10), if it is positive and set $T_i = 0$ otherwise. Intuitively this makes sense. If Red were to move into the Green Zone in Iraq, his initial probability of detection would be so high that he would immediately move to another location. We can also see that by Blue increasing α_i he forces Red to move more frequently and risk in transit detection more often.

Going back to (2) and using the integral of *G(t)* we get the following:

$$\begin{aligned}\exp\{-G_i(t)\} &= \exp\{-q_i t - \frac{\alpha_i t^2}{2}\} \\ &= \exp\{-\frac{\alpha_i}{2}(t + \frac{q_i}{\alpha_i})^2 + \frac{q_i^{\,2}}{2\alpha_i}\} \\ &= \exp\{\frac{q_i^{\,2}}{2\alpha_i}\}\exp\{-\frac{\alpha_i}{2}(t + \frac{q_i}{\alpha_i})^2\}\end{aligned}$$

Making this substitution into (9) produces the following

$$A_i = \exp\{\frac{q_i^{\,2}}{2\alpha_i}\}\int_0^T \exp\{-\frac{\alpha_i}{2}(t + \frac{q_i}{\alpha_i})^2\}dt + \exp\{-G_i(T)\}V_i \tag{11}$$

By letting $u = (t + \frac{q_i}{\alpha_i})\sqrt{\frac{\alpha_i}{2}}$ (10) becomes

$$A_i = \exp\{\frac{q_i^{\,2}}{2\alpha_i}\}\sqrt{\frac{2}{\alpha_i}}\int_{U_1}^{U_2} \exp\{-u^2\}dt + \exp\{-G_i(T)\}V_i \tag{12}$$

Where our upper and lower limits of integration becoming

$$U_{1i} = \frac{q_i}{\sqrt{2\alpha_i}} \quad \text{and } U_{2i} = \frac{q_i}{\sqrt{2\alpha_i}} + T_i\sqrt{\frac{\alpha_i}{2}} \tag{13}$$

Keeping in mind the error function $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\{-y^2\} dy$ we can then obtain A_i in terms of it.

$$A_i = \frac{\exp\{\frac{q_i^2}{2\alpha_i}\}\sqrt{\pi}[erf(U_{2i}) - erf(U_{1i})]}{\sqrt{2\alpha_i}} + \exp\{-G_i(T_i)\}V_i \tag{14}$$

2. Direct Detection

We define direct detection as the threat Red faces, if he is located on the same route as Blue. In reality, Blue can directly search on multiple routes, but for this model Blue's direct search is limited to the route the convoy is on. Clearly, if Red happens to be on the same route as Blue, he will face two risks: 1) the risk of being "given up" and 2) the risk of being found by Blue before he can ambush the convoy. The probability that Red will be detected in a direct search by time t is given by

$$R(t) = 1 - \exp\{-F(t)\} \tag{15}$$

We make the assumption that Blue's level of aggressiveness associated with directly finding Red will be the same regardless of which route Blue is on. If the physical characteristics of the route or other limitations violate this assumption, we will need to differentiate this level of aggressiveness as we did in the indirect detection parameter **α**. Keeping this in mind, we define the forcing function for the risk of direct detection as the following

$$\begin{aligned} f(t) &= p + \beta t \\ F(t) &= \int_0^t f(s) ds = \text{t}p + \frac{\beta t^2}{2} \end{aligned} \tag{16}$$

where p is the initial probability that Red will be found, if he is on the same route as Blue's recon and β (Blue's direct aggressiveness) is the rate at which that risk increases, as long as he remains on the same route as Blue.

Assuming that Blue and Red are on the same route, we will define the time at which Red decides to move as D_i with his expected survival time after leaving route *i* still as V_i. Red's total expected survival time, including the time he spends on the same route as Blue is therefore given by

$$B_i = \int_0^D tf_i(t)\exp\{-F_i(t)\}dt + D_i \exp\{-F_i(D_i)\} + \exp\{-F_i(D_i)\}V_i \tag{17}$$

Further refining this, as we did in the case with indirect detection, we obtain the following

$$B_i = \exp\{-F_i(D_i)\}V_i + \frac{\sqrt{\pi}\exp\{\frac{p_i^2}{2\beta}\}[erf(W_{2i}) - erf(W_{1i})]}{\sqrt{2\beta}} \tag{18}$$

And, as before, our limits of integration become

$$W_{1i} = \frac{p_i}{\sqrt{2\beta}} \text{ and } W_{2i} = \frac{p_i}{\sqrt{2\beta}} + D_i\sqrt{\frac{\beta}{2}} \tag{19}$$

In the same manner as we derived T_i (10), we get the following result for when Red will leave the route he is on, if Blue is directly searching there

$$D_i = \frac{1 - p_i V_i}{\beta V_i} \tag{20}$$

We can make the reasonable assumption that the indirect probability *q* will be less than the direct detection probability *p*. Here we make the assumption that the rate of increase in a direct search, β, will be equal or greater than the rate of increase in a indirect search, α. Put another way, Blue's patrols will be more aggressive and therefore more likely to find Red than any effort to have a third party uncover Red's location. (There may be cases where the inhabitants of an area will be more effective than Blue at capturing Red, but then Red would avoid these areas, as they give him no advantage.) Since $p > q$ and $\beta \geq \alpha$ it is apparent that $D_i < T_i$ and Red will always depart more quickly, if Blue is directly searching on the same route as him. As with T_i, D_i has the potential of being negative. In that event, we will let $D_i = 0$.

3. Stochastic Process

As noted by Owen and McCormick, this is clearly a stochastic game in which Red's survival time is dependent upon how often he is allowed to move. To denote this, we will use A_i^m where *m* represents the number of times Red is allowed to move and *i* is the route that he starts in and the assumption is that Blue is searching on another route. If Blue is on the same route as Red, we will represent Red's expected survival time as before with B_i^m. After he moves, Red will be able to move *m-1* additional times and he expects his remaining survival time to be V_j^{m-1}.

Considering the case when Red is not allowed to move we let $V_j^{-1} = 0$. Red will stay on the route until he is captured. In this case, *T* becomes infinite and our upper bound on equation (12) goes to infinity along with *G(x)*. This leads $Erf(\infty) = 1$ and $\lim_{x \to \infty} e^{-x} = 0$ and equation (14) becomes

$$A_i^0 = \frac{\exp\{\frac{q_i^2}{2\alpha_i}\}\sqrt{\pi}[1 - erf(U_{1i})]}{\sqrt{2\alpha_i}} \tag{21}$$

If we carry this out recursively we see that the general form of (21) becomes

$$A_i^m = \frac{\exp\{\frac{q_i^2}{2\alpha_i}\}\sqrt{\pi}[erf(U_{2i}) - erf(U_{1i})]}{\sqrt{2\alpha_i}} + \exp\{-G_i(T_i)\}V_i^{m-1} \tag{22}$$

Applying this to the time when Red departs the route we get

$$T_i = \max(\frac{1 - q_i V_i^{m-1}}{\alpha_i V_i^{m-1}}, 0) \tag{23}$$

Similarly, we derive B_i^m in the following manner

$$B_i^m = \frac{\sqrt{\pi}\exp\{\frac{p_i^2}{2\beta}\}[erf(W_{2i}) - erf(W_{1i})]}{\sqrt{2\beta}} + \exp\{-F_i(D_i)\}V_i^{m-1} \tag{24}$$

with

$$D_i = \max(\frac{1 - p_i V_i^{m-1}}{\beta V_i^{m-1}}, 0) \tag{25}$$

Finding the expected time until capture along each route, assuming Red is not allowed to move (i.e., m = 0), is relatively simple. For $m > 0$ we must take into consideration the time Red expects to gain from moving. This is done by calculating a the expected time till detection for the route he is planning to move to by the probability that he will successfully complete the move.

As Red moves from route i to route j, there is the probability ρ_{ij} that he will complete his move without being captured. (We can think of P as being a symmetric matrix made up of these probabilities based on the distances between routes with zeros along the diagonal.) Red will then survive an additional A_j or B_j when he arrives. If Red moves to a route where Blue is not directly searching, he can be expected gain an additional τ_{ij} units of time if he survives the move

$$\tau_{ij} = \rho_{ij} A_j \tag{26}$$

Similarly, if Red moves to a route where Blue is directly searching he can expect to gain σ_{ij} units of time if he survives the move

$$\sigma_{ij} = \rho_{ij} B_j \tag{27}$$

Focusing on a single route I, we can see the expected gain in survival time Red may obtain from moving to a different route j. By ordering our τ_j in decreasing order (i.e. $\tau_1 \geq \tau_2 \geq ... \geq \tau_n)$, we can use the following algorithm developed by Owen and McCormick to determine the increase in survival time after the move, V_i , from route i.

$$L_k = \sum_{j=1}^{k} \frac{1}{\tau_j - \sigma_j} \quad \text{and } V_k = \frac{\sum_{j=1}^{k} \frac{\tau_j}{\tau_j - \sigma_j} - 1}{L_k} \tag{28}$$

Algorithm.

1. Let k = n (the number of routes)
2. Let $v = v_k$, computed using (28)
3. If $v \leq \tau_k$, proceed to step 6
4. If $v > \tau_k$, let k=max $\{ j \mid \tau_j > v \}$
5. Return to step 2
6. Compute **x*** and **y*** using (29)

Solving for (A_i^0, B_i^0) then V_i^0 allows us to solve for (A_i^1, B_i^1), D_i^1, T_i^1 then V_i^1. Doing this iterative process allows us to find the expected times until capture for increasing values of *m*, the number of times we allow Red to change routes.

Owen and McCormick discovered that while computing these quantities for increasing values of *m* their values change very little after approximately *m* = 10. In their paper they proved convergence with the assumption that some risk is incurred every time Red moves (i.e. $\rho_{ij} < 1$). The greater the ρ_{ij}, the faster the expected times converge. In a risky environment, it should not be necessary to compute for values of *m* greater than 10 as it is unlikely Red can look more than 5 or 6 steps in advance to see where he should move.

Coincidently, if we were just concerned with Red's attempt at survival, we can compute the optimal strategies for Red's use of routes and Blue's strategy for finding him. Given that the game has a value vk, we can compute x^*, y^*, the optimal strategies for Red and Blue respectively, in the following manner

$$\begin{aligned} x_i^* &= \frac{\tau_i - v_k}{\tau_i - \sigma_i} \quad \text{for } 1 \le i \le k, \text{ and } 0 \text{ for } k+1 \le i \le n \\ y_i^* &= \frac{\mathrm{L}_k}{\tau_i - \sigma_i} \quad \text{for } 1 \le i \le k, \text{ and } 0 \text{ for } k+1 \le i \le n \end{aligned} \tag{29}$$

A benefit of this model is its ability to determine where Red is most likely to move to when he does decide to move. Owen and McCormick further explored this property with the assumption that we know the last location Red occupied [3]. We will not pursue the same analysis in this model. If we knew that Red moved, and from which route, it gives us a route to use without fear of ambush, thus invalidating the need for further analysis for route selection. It is worthy to note this property in the event Blue does learn of Red's last position and wishes to send a patrol out to uncover him.

B. ANALYSIS OF SEARCH MODEL

1. Decreasing Expected Time to Detection for a Single Route

We make the assumption that Red can move multiple times to both increase his chances of attacking Blue and to avoid detection. Therefore, the case A_i^0 and B_i^0 do not concern us other than to establish the semi-steady state for A_i^m and B_i^m. Blue's goal is to drive T_i and D_i to zero in order to make that route unattractive to Red and safer for the convoy. Keeping in mind equations (23) and (25), simply increasing the Blue's indirect aggressiveness (α_i) or direct aggressiveness (β) will decrease Red's expected time to detection but it will not get it to zero. The ultimate goal is to drive the values $q_i V_i^{m-1} \geq 1$ and $p_i V_i^{m-1} \geq 1$. Doing so brings Red's departure times to zero, and he moves as soon as he finds himself on that route. Blue can accomplish this by placing all of his effort onto the route with the largest initial probability of indirect or direct detection. Since we are dealing with a single route, we will focus on direct detection from here on. Since we are dealing with only one route the value then becomes B^0 (keeping in mind that $V^{-1} = 0$ and letting k=1 for equation (26)) with the additional assumption that Red will be on that route to ambush Blue and neither will switch to another route. Blue is then left with trying to drive B^0 to zero. How aggressive must the patrols be (β) to make this happen? Taking equation (21) and applying it to B^0 gives us

$$B^0 = \frac{\exp\{\frac{p_i^2}{2\beta}\}\sqrt{\pi}[1 - erf(W_{1i})]}{\sqrt{2\beta}} \tag{30}$$

Evaluating (27) shows us that B^0 can never be zero since β, p_i are always positive. The only way to reduce the value of the route is to increase Blue's direct aggressiveness (β) and the payoff for this effort decreases exponentially. This is what we have come to intuitively understand and, although we are not limiting ourselves to a single route, it can help us see our diminishing rate of return on effort along a single route. From this foundation we will shift our analysis to multiple routes.

2. Decreasing Expected Time to Detection for Multiple Routes

As with Ruckle's work on the geometric approach to the game [2], we will assume Blue takes a single route to his destination (i.e., straight line). We will also make the assumption that the routes are independent of each other (i.e., they do not cross). This is important, because if the routes intersected at a common point, then it becomes a game with one route at that point. We can then study the routes independent of each other.

Using the above model, what must Blue do to secure a route? Clearly if he is very aggressive (both indirectly and directly (α_i and β)) he can then bring all A_i and B_i close to zero, but at the expense of spending greater resources. We have already established that they can never be zero because of the exponential nature of the risk. The best Blue can do is to drive the time at which Red will depart a route (T_i and D_i) to zero so that Red will move immediately away from that route if he is on it. If he is able to accomplish this, then the equations (22) and (24) become:

$A_i^m = \exp\{-q_i\}V_i^{m-1}$ and $B_i^m = \exp\{-p_i\}V_i^{m-1}$

Let us first focus on how Blue might get these departure times (T_i and D_i) to be zero. By increasing his indirect aggressiveness (α), Blue can bring expected survival time, if Blue is not on the route (*A*) closer to the expected survival time if Blue was on the route (*B*) and the value of that route goes to *B*. As the value of route is more dependent on *A*, Blue gets a greater payoff in the reduction of the route's value by being indirectly aggressive (α) but this does not get Red's expected survival time any lower than if Blue was on that route (*B*). To reduce Red's expected survival time more, and ultimately to get it to zero, Blue must focus his effort on directly finding Red. This suggests that Blue must dramatically increase his direct aggressiveness (β) for this to happen. However, Blue has another option.

As already mentioned the value of a route, in the presence of other routes, (V_i^{m-1}) must be significantly large enough so that $q_i V_i^{m-1} \geq 1$ and/or $p_i V_i^{m-1} \geq 1$. To do this, Blue simply has to add routes under consideration. However, as Blue adds routes and increases V_i^{m-1} the value of Red's expected survival time with Blue on the route (B_i^m) goes up. To strike a balance between the two, we find that $V_i^{m-1} = \frac{1}{q_i}$ provides us the minimum value we need a route to be so as to bring the time when Red moves (D_i) to zero (likewise for T_i and $V_i^{m-1} = \frac{1}{q_i}$). To minimize the expected survival time for Red, if Blue is on the route (B_i^m), Blue must now adjust his efforts too so that $V_i^{m-1} = \frac{1}{q_i}$ and no more. In reality, Blue will want to provide enough "useable" routes where Red will move immediately, if he discovers that Blue is on them (i.e., $D_i = 0$) while accepting the increase in B_i^m so that he can randomize which route he takes. For the purposes of this game, we will only consider those routes that Blue is considering using and is currently exerting effort (either indirect or direct) to find Red. (Remember that *F (t)* and *G (t)* must be strictly increasing). In other words, Blue wants to provide enough viable routes for Red to choose from and hide on while reducing the attractiveness of certain routes. This may not always be possible, since Blue often has only a finite number of routes to choose from and must make the most of what is available. In addition to limited routes, the enemy also gets a vote. Red, being an intelligent player, will try to overcome Blue's efforts to avoid him by applying his own ambush model.

III. AMBUSH MODEL

A. OVERVIEW

No matter how unattractive Blue makes a route, if he continues to utilize that route, Red will be tempted to move onto that route and ambush Blue. Red will risk direct detection to achieve his own goals. Blue's probability of encountering a hazard, either indirect or direct, as he continues to utilize the same route convoy after convoy will increase. We assume that Red's efforts to intercept Blue will remain constant regardless of route and represent it with the variable γ. If this assumption is not valid, we will have to differentiate Red's aggressiveness by route, as we did for **α**. The more aggressive Red becomes the greater the value of γ.

1. Indirect Hazard

Every convoy runs the risk of not completing the journey regardless of enemy action. This could be the result of treacherous terrain (think of Hannibal's journey across the Alps) or a longer route that can result in more breakdowns. With this in mind, we define r_i as our initial probability of success for Blue crossing route *i*. Note that this model takes the same form as our model for Red's threat of indirect detection. As we did there, we will start by defining *S(t)* to be the probability of an unsuccessful completion of a convoy.

$$S(t) = 1 - \exp\{-H_i(t)\} \tag{31}$$

where our strictly increasing risk is defined by

$$\begin{aligned} h_i(t) &= r_i + \gamma t \\ H_i(t) &= \int_0^t h_i(s)ds = tr_i + \frac{\gamma t^2}{2} \end{aligned} \tag{32}$$

Following the same derivation we have used before, the expected survival time for Blue along route *i*, assuming he can change routes *m* times, is:

$$C_i^m = \frac{\exp\{\frac{r_i^2}{2\gamma}\}\sqrt{\pi}[erf(K_{2i}) - erf(K_{1i})]}{\sqrt{2\gamma}} + \exp\{-H_i(M_i)\}V_i^{m-1} \tag{33}$$

where the limits of integration are given by

$$K_{1i} = \frac{r_i}{\sqrt{2\gamma}} \quad \text{and } K_{2i} = \frac{r_i}{\sqrt{2\gamma}} + M_i\sqrt{\frac{\gamma}{2}} \tag{34}$$

and the expected time of departure from route *i* is

$$M_i = \max\left(\frac{1 - r_i V_i}{\gamma V_i}, 0\right) \tag{35}$$

2. Direct Hazard

Different routes provide Red with a greater advantage of ambushing Blue. Some routes provide more than adequate cover for Red to hide or choke points for him to utilize. Given this condition, we will assign the initial probability of a successful ambush to each route as s_i and our strictly increasing risk is defined by

$$\begin{aligned} j_i(t) &= s_i + \gamma t \\ J_i(t) &= \int_0^t j_i(z)dz = ts_i + \frac{\gamma t^2}{2} \end{aligned} \tag{36}$$

This leads us to define our expected time of survival for Blue on route *i* as

$$E_i^m = \frac{\exp\{\frac{s_i^2}{2\gamma}\}\sqrt{\pi}[erf(L_{2i}) - erf(L_{1i})]}{\sqrt{2\gamma}} + \exp\{-J_i(N_i)\}V_i^{m-1} \tag{37}$$

where the limits of integration are given by

$$L_{1i} = \frac{s_i}{\sqrt{2\gamma}} \quad \text{and } L_{2i} = \frac{s_i}{\sqrt{2\gamma}} + N_i\sqrt{\frac{\gamma}{2}} \tag{38}$$

and the expected time of departure from route *i* is

$$N_i = \max\left(\frac{1 - s_i V_i}{\gamma V_i}, 0\right) \tag{39}$$

3. Stochastic Process

We use the same method as we did for the Search Model in developing the expected time to ambush for Blue on each route given that Blue can change routes *m* times. If $m = 0$, then our expected times for indirect or direct hazard respectively are:

$$C_i^0 = \frac{\exp\{\frac{h_i^2}{2\gamma}\}\sqrt{\pi}[1-erf(L_{1i})]}{\sqrt{2\gamma}} \text{ and } E_i^0 = \frac{\exp\{\frac{j_i^2}{2\gamma}\}\sqrt{\pi}[1-erf(K_{1i})]}{\sqrt{2\gamma}} \tag{40}$$

As with Red, Blue will face some risk by changing routes such as a greater distance to cover. We will continue to use the terms *tau* and *sigma* to represent the expected gain in time by moving onto route *j*, depending on whether Red is present or not (respectively), but we will use Θ to represent our matrix of completion probabilities, as we did for P in the previous model.

If Blue moves to a route where Red is not directly searching, he can expect to gain an additional τ_{ij} units of time if he survives the move

$$\tau_{ij} = \theta_{ij} C_j \tag{41}$$

Similarly, if Blue moves to a route where Red is directly searching he can expect to gain σ_{ij} units of time if he survives the move

$$\sigma_{ij} = \theta_{ij} E_j \tag{42}$$

Then by using (28) we can determine the value of each route. Using this we can iteratively find the expected survival time along each route by increasing *m* until we see a stable time appear while using equations (33) and (37). We should keep in mind that the computation of the expected survival time after leaving route *i* (*V*) is done as before in equation 28 with the expected times till indirect hazard being sorted in decreasing order (i.e. $\tau_1 \geq \tau_2 \geq ... \geq \tau_n$). If we wanted to see what Red and Blue's optimal strategies are in the absence of the search model, we can use equation 29. Note, however, that in this case x^* is Blue's optimal strategy, since he is the row player and y^* is Red's optimal strategy, since he is the column player.

B. ANALYSIS OF AMBUSH MODEL

1. Increasing Expected Survival Time on a Single Route

As we saw in the analysis of the Search Model, restricting ourselves to a single route reduces the value of the game to that of the expected survival time associated with a direct hazard. Again, this value can never be zero but gets exponentially closer to zero with increasing levels of effort (γ) on the part of Red to increase the probability of hazard. The challenge for Blue now, if he wishes to make it through the route successfully, is to ensure his expected survival time, E^0, is greater than Red's expected survival time, B^0, along that route. He can do this by making sure Red's probability of detection, p, is greater than his initial probability of a hazard, r, and that his efforts to increase the rate of detection, β, are greater than Red's efforts to increase the linear rate of interference, γ.

Of course, intuitively, this is what one expects. If the route is a highway with clear fields of view, and Blue actively sends recons up and down the route, then he is apt to survive longer along that route than Red. The converse is also intuitive. If a route goes through an area where Red can easily hide, and Red is very aggressive along that route, then Blue's survival time will be lower than that of Red. Give the two options, Blue should always choose the former; thus leaving Red only the option of significantly increasing his efforts γ to bring E^0 closer to B^0. Given enough time though, Blue will be ambushed along that route. Blue can reduce this risk by adding routes to choose from. This is especially true if the routes are not significantly favorable to Blue.

2. Increasing Expected Survival Time on Multiple Routes

In this model, Blue's only influence over his expected time of survival until encountering a hazard is to add more routes under consideration. As we see in equations (33) and (37) the expected survival time after moving goes up as we add more routes. However this also drives some of the times till departure, (35) and (39), to zero as some routes become more favorable. As with the Search Model, it becomes disadvantageous for Blue to add more routes, if they do not offer an advantage to routes already under consideration. There is also the practical matter of having only a finite number of possible routes to choose from in a realistic scenario. We will therefore limit our study to a few routes with the understanding that Blue has chosen from the best available to him.

IV. BIMATRIX MODEL

A. OVERVIEW

We have now developed two models. In each model, one player is trying to maximize his survival time while minimizing the other player's survival time. Taking these two models we will define the payoff matrices in the following manner

Ψ is the Search Payoff Matrix where $\psi_{ij} = \{ \begin{smallmatrix} A_{ij} & i \neq j \\ B_{ij} & i = j \end{smallmatrix}$

Λ is the Ambush Payoff Matrix where $\lambda_{ij} = \{ \begin{smallmatrix} C_{ij} & i \neq j \\ E_{ij} & i = j \end{smallmatrix}$ (43)

(note: From here on we'll use the transpose of Λ since we need Blue to play the columns of both matrices, but Λ was formed with Blue playing the rows)

These payoff matrices form the basis for our bimatrix game. John Nash's renowned paper on non-cooperative games [4] in 1951 proved that for every bimatrix game a pair of strategies exist that, if played by both players, maximize the value for both. At this equilibrium point, neither player can obtain a greater value by applying a different strategy while his opponent's strategy remains unchanged. If both players change their strategies, then either party, or both, can obtain a greater value for their game..

Both Red and Blue can choose a single route (a pure strategy) or they can randomize their route choice by assigning a probability to the likelihood that they will use it (a mixed strategy). Let X be the set of all possible mixed strategies for Red and Y be the set of all possible mixed strategies for Blue. The expected survival time for Red is $E_{\mathrm{Re}d}(x,y) = x^T \Psi y$ for some $x \in X$ and $y \in Y$ and likewise for Blue the expected survival time is $E_{Blue}(x,y) = x^T \Lambda^T y$ for some $x \in X$ and $y \in Y$. The Nash equilibrium is the pair of mixed strategies that maximize the survival time for both players. Letting x^* and y^* be our optimal strategies we can state it in the following way

$$E_{\mathrm{Re}d}(x^*, y^*) = x^{*T} \Psi y^* \geq \mathrm{x}^T \Psi \mathrm{y}^* \quad \forall \ \mathrm{x} \in \mathrm{X}$$
$$E_{Blue}(x^*, y^*) = x^{*T} \Lambda^T y^* \geq \mathrm{x}^{T*} \Lambda^T \mathrm{y} \quad \forall \ \mathrm{y} \in \mathrm{Y}$$

Once we find x^* and y^* we define the value of the game for each player as $V_{Red} = x^{*T}\Psi y^*$ and $V_{Blue} = x^{*T}\Lambda^T y^*$. The player with the larger value has the advantage given the Nash equilibrium identified. Unfortunately, there is not always a single equilibrium point in their available strategies.

1. Finding Pure Equilibrium Points

Finding pure equilibrium points is relatively easy. Looking at Red's payoff matrix, Ψ, we choose the largest expected survival time in each column. For Blue, we look at the transpose of his payoff matrix, Λ^T, and choose the largest value in each row. the locations (i,j) where these locations occur simultaneously are called the pure strategies where Red will use route i and Blue will use route j. As an example, take the following payoff matrices

$$\Psi = \begin{bmatrix} 2 & \langle 5 \rangle & 5 \\ \langle 6 \rangle & 3 & \langle 6 \rangle \\ 4 & 4 & 1 \end{bmatrix} \quad \Lambda^T = \begin{bmatrix} 10 & 16 & \langle 18 \rangle \\ \langle 22 \rangle & 8 & 18 \\ \langle 22 \rangle & 16 & 12 \end{bmatrix}$$

We have labeled using < > those column and row entries that are the largest for Red's columns and Blue's rows (respectively). From the example, we see that our equilibrium point is met when Blue chooses route 1 and Red chooses route 2. The advantage is clearly in Blue's favor, as he is expected to survive longer than Red. This equilibrium should not be any surprise, as our payoff matrices are constructed in such a fashion that the pure strategy for either player will always be the route that provides the longest possible expected survival time, if the adversary is not on the route. Furthermore, since $A_i \geq B_i$ and $C_i \geq D_i$ the only possibility of Red or Blue choosing the same route (equilibrium on the diagonal) is in the event both indirect and direct detection/hazard times are equal. We can intuitively understand this result, however unlikely, as both Red and Blue have nothing to gain if they end up on the same route.

If Red or Blue wanted to alter the expected survival times, they could do so by spending resources (increasing their aggressiveness) to increase the rate of detection or hazards $(\alpha, \beta$ and $\gamma)$ thereby decreasing the value for their opponent.

We must note that more than one pure strategy may appear using the following method. The strategy to choose would be the one that offers the maximum survival time to both players. If such a strategy does not exist, or multiple pure strategies occur with the same value, Red and Blue will need to determine their optimal mixed strategies. This is far more realistic, since Blue and Red will not always play a perfect game ensuring that they never pick the same route. Blue and Red will want to randomize their route selection as to not give an advantage to the other player in detecting them.

2. Finding Mixed Equilibrium Points

Finding all possible mixed Nash equilibria can be a daunting task. In 1964, Lemke and Howson [5] showed how to obtain all of the mixed equilibrium points in a two person game using non-linear programming. Their algorithm states that the strategies x^* and y^* are Nash equilibria, if and only if, they maximize the following non-linear equation and constraints [6]:

$$\begin{aligned}
&\max_{x,y,p,q} x^T \Psi y + x^T \Lambda y - p - q \\
&\text{subject to:} \\
&\Psi y \le pJ_n \quad \Lambda x \le qJ_n \text{ (where } J_n \text{ is a } n \times 1 \text{ vector of all ones)} \\
&x_i \le 0 \text{ and } y_i \le 0 \ \forall i \in (1..n) \\
&\sum_{i=1}^{n} y_i = \sum_{j=1}^{n} x_j = 1 \\
&(p^* = V_{\mathrm{Re}d} \text{ and } q^* = V_{Blue})
\end{aligned} \tag{44}$$

Solving the above problem is best done using software. Barron [6] provides the Maple and Mathematica commands for setting up and solving such a problem. There are also multiple software packages available, such as SNOPT and KNITRO, which can be used for solving problems involving a large number of routes. For the examples given below with relatively few routes, we will rely on Maple's NLPSolve command to find multiple mixed Nash Equilibria. We accomplish this by altering the starting point for the non-linear program search by varying the values of *p* and *q*. This is no trivial task as the upper bound on number for equilibria is theorized to be $2^n - 1$ in an *n* x *n* game [7].

In a perfect world, Blue and Red will agree to use a pure equilibrium, and they will never meet on a route. Both will choose the routes that provide them the greatest indirect detection time without being on the route together. This is unlikely as each will be inclined to use this predictability to their advantage and actively intercept the other. A better way to approach this problem is to determine the value of the individual games (search or ambush) and use these as our starting points for determining the mixed equilibrium strategies (i.e., *p* and *q* in 44). We can determine the expected payoffs of the individual games in the following way

$$
\begin{aligned}
V_{Search} &= x^{*T}_{\mathrm{Re}d-Search} \Psi y^{*}_{Blue-Search} \\
V_{Ambush} &= x^{*T}_{\mathrm{Re}d-Ambush} \Lambda^{T} y^{*}_{Blue-Ambush}
\end{aligned}
\tag{45}
$$

These two values, while interesting, do not take into account the dynamics of both players being threatened while simultaneously threatening their opponent. What they do provide is a starting point when we apply non-linear programming to determine both players' optimal strategies when faced with their competing self-interests of survival and attack. Depending on the risk either faces from moving, we may still encounter pure strategies where Red or Blue decide to use a single route rather than run the risk of changing routes even if they can shorten the expected survival time of the other. Clearly, if the risk of movement is not too great, a mixed strategy would best benefit both players as they can randomize their route selection. In the examples that follow, we will explore several variations of this game.

B. EXAMPLES

1. High Risk of Movement for Red and Uniform α

In this example, we will examine the game where Blue may choose from six routes. The situation is such that Red faces a significant risk every time he decides to move—less than 50% probability of successful completion of the move. Blue has relative freedom of movement—greater than 90% probability of successfully changing routes. The rate of increase in the probability of detection is uniform for all players $(\alpha = \beta = \gamma = .01)$.

We list Red and Blue's initial probabilities of detection in the following table, along with the variables assigned to them in Chapters II and III.

Route	p (Search Active)	q (Search Passive)	j (Ambush Active)	r (Ambush Passive)
1	0. 5	0.3	0.4	0.1
2	0.7	0.4	0.6	0.1
3	0.4	0.1	0.3	0.2
4	0.3	0.2	0.4	0.1
5	0.2	0.1	0.5	0.2
6	0.5	0.1	0.3	0.1

Table 1. Example I—Probabilities of Ambush / Detection

Our first step is to determine the expected time to capture / ambush assuming that neither Red or Blue are allowed to change routes (m =0). Doing so using the Search Model gives us

m=0	1	2	3	4	5	6
A_i^0	3.05	2.37	6.56	4.21	6.56	6.56
B_i^0	1.93	1.40	2.37	3.05	4.21	1.93

Table 2. Example I—Red's Expected Survival Time with No Moves

Using this, we can then start to determine the value of the routes and the time Red will stay on each route. To compute how much extra time Red expects to gain from moving from route i to route j, we need to define the risk he faces during the move. Using (26), (27), (41), (42) and Red and Blue's probability of successfully changing routes, given respectively by the matrices P and Θ below, we determine that letting m = 4 we reach stability in that the values for A, B, T, D, and V converge to within the first two decimal places.

P	1	2	3	4	5	6
1	0	0.452	0.409	0.452	0.435	0.402
2	0.452	0	0.452	0.435	0.452	0.435
3	0.409	0.452	0	0.402	0.435	0.452
4	0.452	0.435	0.402	0	0.452	0.409
5	0.435	0.452	0.435	0.452	0	0.452
6	0.402	0.435	0.452	0.409	0.452	0

Table 3. Example I—Red's Probabilities for a Successful Moves

Θ	1	2	3	4	5	6
1	0	0.98	0.96	0.94	0.93	0.92
2	0.98	0	0.98	0.96	0.94	0.93
3	0.96	0.98	0	0.98	0.96	0.94
4	0.94	0.96	0.98	0	0.98	0.96
5	0.93	0.94	0.96	0.98	0	0.98
6	0.92	0.93	0.94	0.96	0.98	0

Table 4. Example I—Blue's Probabilities for a Successful Move

The values of A, T, B, D, and V all correspond to the equations given in Chapter II.

m=4	1	2	3	4	5	6
A_i^4	3.05	2.45	6.56	4.21	6.56	6.56
T_i^4	13.83	1.04	34.97	23.08	40.76	33.76
B_i^4	2.28	2.44	2.38	3.05	4.21	2.29
D_i^4	0	0	4.97	13.08	30.76	0
V_i^4	2.28	2.44	2.22	2.32	1.97	2.29

Table 5. Example I—Red's Expected Survival Time with Multiple Moves

For this example, the optimal strategies for Red and Blue (in the absence of the Ambush model) are

	1	2	3	4	5	6
$x^*_{Red-Search}$	0	0	0.2661	0	0.4734	0.2605
$y^*_{Blue-Search}$	0	0	0.2661	0	0.4734	0.2605

Table 6. Example I—Optimal Search Strategies

Red can expect the following survival time using this strategy: $V_{Search} = x^{*T}_{\mathrm{Re}d-Search}\Psi y^{*}_{Blue-Search} = 5.446$. Note that Red and Blue share the same route selection strategies in this example. This will not always be the case.

We need to now find the expected survival times for Blue using the Ambush model. As with the Search model, we start by determining the initial expected survival times along each route assuming that Blue is not allowed to change routes once one is selected (m = 0).

m=0	**1**	**2**	**3**	**4**	**5**	**6**
C_i^0	3.13	3.13	2.55	4.13	2.55	3.13
E_i^0	1.82	1.39	2.13	1.82	1.58	2.13

Table 7. Example I—Blue's Expected Survival Time with No Moves

Since Blue faces less risk moving from cell to cell, the expected times of survival converge much slower than for Red. By m = 35 we get convergence in the first two decimal places.

Table 8 shows the values of C, M, E, N, and V as given in Chapter III.

m=35	**1**	**2**	**3**	**4**	**5**	**6**
C_i^{35}	4.78	4.80	4.57	4.79	4.53	4.77
M_i^{35}	1.25	1.24	0.19	1.24	0.21	1.25
E_i^{35}	4.45	4.48	4.56	4.47	4.52	4.44
N_i^{35}	0	0	0	0	0	0
V_i^{35}	4.45	4.48	4.56	4.47	4.52	4.44

Table 8. Example I—Blue's Expected Survival Time with Multiple Moves

It should not surprise us that all N are zero. With so little risk to Blue's movements, he will change routes immediately, if he finds himself on the same route as Red. Keep in mind that in this game Blue is playing the rows, and Red is playing the columns of our matrix Λ, however, for the sake of consistency, we will continue to define

Blue's optimal strategy as y and Red's as x. The optimal mixed strategy for each player in the absence of the Search model thus becomes

	1	2	3	4	5	6
$x^*_{\mathrm{Re}d-Ambush}$	0.2519	0.2519	0	0.2519	0	0.2443
$y^*_{Blue-Ambush}$	0.2125	0.3063	0	0.2750	0	0.2062

Table 9. Example I—Optimal Ambush Strategies

The value of the Ambush model is $V_{Ambush} = y^{*T}_{Blue-Ambush} \Lambda x^*_{\mathrm{Re}d-Ambush} = 4.702$, and this also gives us the time Blue can expect to survive without an ambush using the available routes. We note here that $V_{Ambush} < V_{Search}$, and given these results alone, we expect an ambush to occur before Red is discovered. In this event, Blue may want to find additional routes or find a way to increase the rate of indirect detection in those cells still relevant for Red to use.

Using the Search Model, we can vary the values for α and β to see the effect of each. The graphs below show the change in V_{Search}, as we vary these parameters while keeping the other one constant.

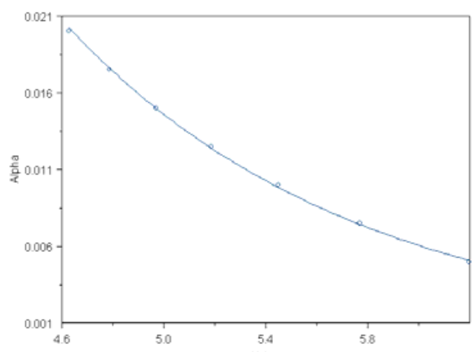

Figure 1. α varies while β=.01

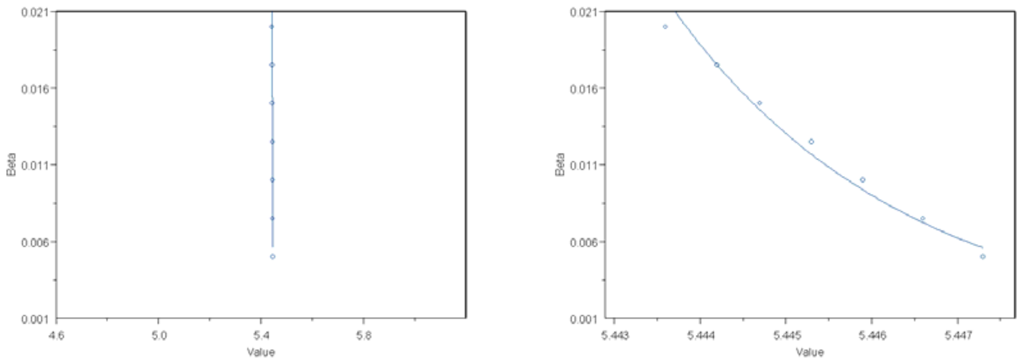

Figure 2. β Varies while α=.01 (Same Graph – Different scales for horizontal axis)

Clearly, Blue's indirect efforts to detect Red (**α**) reduce Red's expected survival time (***V***$\boldsymbol{V_{search}}$) with greater efficiency than Blue's direct efforts (**β**). As expected, as both direct and indirect efforts increase they show a decreasing return on their ability to reduce Red's expected survival time. If Blue finds himself at a disadvantage, (i.e., his expected survival time is less than Red's) this analysis can help the commander decide where to place his efforts (direct or indirect) to get the best reduction in Red's expected survival time.

Taking both games into consideration, we obtain the following bi-matrix consisting of Ψ and Λ^T; each cell contains a value from each individual payoff matrix (ψ, λ).

	1	**2**	**3**	**4**	**5**	**6**
1	(2.28, 4.45)	(3.05, 4.80)	(3.05, 4.57)	(3.05, 4.80)	(3.05, 4.53)	(3.05, 4.77)
2	(2.45, 4.78)	(2.44, 4.48)	(2.45, 4.57)	(2.45, 4.80)	(2.45, 4.53)	(2.45, 4.77)
3	(6.56, 4.78)	(6.56, 4.80)	(2.38, 4.56)	(6.56, 4.80)	(6.56, 4.53)	(6.56, 4.77)
4	(4.21, 4.78)	(4.21, 4.80)	(4.21, 4.57)	(3.05, 4.48)	(4.21, 4.53)	(4.21, 4.77)
5	(6.56, 4.78)	(6.56, 4.80)	(6.56, 4.57)	(6.56, 4.80)	(4.21, 4.52)	(6.56, 4.77)
6	(6.56, 4.78)	(6.56, 4.80)	(6.56, 4.57)	(6.56, 4.80)	(6.56, 4.53)	(2.29, 4.44)

Table 10. Example I—Bimatrix Model

We immediately note that there are multiple pure Nash Equilibrium points in the above matrix. The points that yield the maximum survival time for both are outlined in black above. If Blue and Red were purely rational players, they would avoid each other completely. Blue would choose either route 2 or 4 and Red would choose 3, 5 or 6. Blue would then obtain a value of 4.80 and Red a value of 6.56 for the bimatrix game. Clearly Red has the advantage in this scenario. To find the mixed Nash Equilibrium points we resort to non-linear optimization.

Why do we need to find mixed equilibrium, if we already have several pure strategies from which to choose? If Red and Blue collaborate to ensure they both avoid capture/ambush for the longest possible time, then pure strategies are the answer. However, both parties can be less concerned about their own survival time and more concerned with reducing the survival time of their adversary. We call these different strategies as risk adverse (wishing to maximize one's own survival time) and risk prone (disregarding self-preservation in an effort to reduce the others). By adjusting the initial point in our non-linear programming, we arrive at different optimal mixed strategies. In the tables below, we declare our starting point as (*p*, *q*) where *p* is Red's payoff from the Search game and *q* is Blue's payoff from the Ambush game, refer to equation (44).

We assume that each player wishes to find the optimal strategy that gets them as close to the value of their individual games as possible. This means we conduct our non-linear optimization from the initial value of (5.446, 4.702). Using a software package (in this case MAPLE's NLPSolve command) we obtain the following mixed strategies:

(5.446,4.702)	**1**	**2**	**3**	**4**	**5**	**6**
x^*_{Red}	0	0	0.283	0	0.717	0
y^*_{Blue}	0	1	0	0	0	0

Table 11. Example I—Optimal Bimatrix, Risk Adverse Strategies

Our players are now using strategies that are consistent with the pure strategies previously noted (and only Red is using a mixed strategy). Note that they still do not choose routes that intersect with their adversary. Using these mixed strategies, we can determine the value of the game for each player as: $V_{\mathrm{Red}} = x^{*T}\Psi y^* = 6.56$ and $V_{Blue} = x^{*T}\Lambda y^* = 4.80$. This should not be surprising as the pure strategies all lead to the same values of the game for each player. In practical applications, any mixed strategy involving the pure strategies will yield the same result. Over the course of time, each player is benefited by randomizing their choices so a mixed strategy is preferable over a pure strategy, especially when they lead to the same values for the game.

What if each player was so aggressive that he was not concerned with maximizing his own survival time? By setting our initial point to (*0,0*) we get the following strategies:

(0,0)	**1**	**2**	**3**	**4**	**5**	**6**
$x^*_{\mathrm{Re}d}$	0	0	0.283	0	0.717	0
y^*_{Blue}	0	.643	0	.357	0	0

Table 12. Example I—Optimal Bimatrix, Risk Prone Strategies

Blue has now adopted a mixed strategy and Red's strategy has not changed. The values of the games are unchanged at: $V_{\mathrm{Re}d} = x^{*T}\Psi y^* = 6.56$ and $V_{Blue} = x^{*T}\Lambda y^* = 4.80$. Clearly there are multiple strategies for Red and Blue that lead to the same values. In reality, a route that is conducive to a successful ambush (choke points with lots of cover) is also conducive to hiding. Likewise, a route that is favorable for a convoy (wide open spaces) is not favorable for the enemy seeking to avoid detection. Therefore, it should not be surprising that Red and Blue seek out different routes.

Blue can use this information to his advantage by choosing his route strategy among the mixed strategies among routes 2 and 4 that lead to $V_{Blue} = 4.80$ while avoiding routes 3 and 5. Blue also has the benefit of learning the mixed strategy Red will adopt in the Nash Equilibrium. It is important to note that if Blue was to use this information to send a separate patrol to find Red using this strategy, it would violate the assumptions set forth at the beginning of this paper. The game would then become one of 3 players (vs. 2). Next, we will see what happens when Red is allowed to move with less risk.

2. Low Risk of Movement for Red and Uniform α

In this example, we will set the probabilities of detection/hazard for Red and Blue (indirect and direct) very close to each other. This will drive them to consider the same routes. We will also make it less likely that Red will be intercepted in transit. As in Example I with Blue, both players will have a 90% or greater probability of successfully changing routes. The linear rate of increase in the probability of detection/hazard is uniform for all players $(\alpha = \beta = \gamma = .01)$.

Red and Blue's initial probabilities of detection are given in the following table along with the variables assigned to them in Chapters II and III.

Route	p (Search Active)	q (Search Passive)	j (Ambush Active)	r (Ambush Passive)
1	0.5	0.3	0.4	0.2
2	0.5	0.4	0.6	0.1
3	0.4	0.2	0.4	0.2
4	0.3	0.2	0.3	0.1
5	0.3	0.1	0.2	0.1
6	0.2	0.1	0.3	0.1

Table 13. Example II—Probabilities of Ambush / Detection

As before, our first step is to determine the expected time to capture / ambush assuming that neither Red or Blue are allowed to change routes (m =0). Doing so using the Search Model gives us

m=0	1	2	3	4	5	6
A_i^0	3.05	2.37	63.56	4.21	6.56	6.56
B_i^0	1.93	1.40	2.37	3.05	4.21	1.93

Table 14. Example II—Red's Expected Survival Time with No Moves

Once again, we go through the process outlined in Example 1 to determine the final values for each route. Both Blue and Red face little risk in moving as given inTable 15. Our equations (26), (27), (41) and (42) converge much slower, and we must calculate larger values of m before reaching a stable solution. As with Blue in the first example, we will need m = 35 to get convergence to the first two decimal places.

P and Θ	1	2	3	4	5	6
1	0	0.98	0.96	0.94	0.93	0.92
2	0.98	0	0.98	0.96	0.94	0.93
3	0.96	0.98	0	0.98	0.96	0.94
4	0.94	0.96	0.98	0	0.98	0.96
5	0.93	0.94	0.96	0.98	0	0.98
6	0.92	0.93	0.94	0.96	0.98	0

Table 15. Example II—Red and Blue's Probabilities for a Successful Move

The values of A, T, B, D, and V are found with the equations given in Chapter II.

m=35	1	2	3	4	5	6
A_i^{35}	6.86	6.96	7.49	7.13	7.55	7.52
T_i^{35}	0	0	4.51	0	4.29	4.34
B_i^{35}	6.86	6.96	6.89	7.13	7.00	6.95
D_i^{35}	0	0	0	0	0	0
V_i^{35}	6.86	6.96	6.89	7.13	7.00	6.95

Table 16. Example II—Red's Expected Survival Time with Multiple Moves

For this example, the optimal strategies for Red and Blue (in the absence of the Ambush model) are

	1	2	3	4	5	6
$x^*_{\mathrm{Re}d-Search}$	0	0	0.3206	0	0.3459	0.3335
$y^*_{Blue-Search}$	0	0	0.2673	0	0.3992	0.3335

Table 17. Example II—Optimal Search Strategies

Red can expect the following survival time using this strategy: $V_{Search} = x^{*T}_{\mathrm{Re}d-Search}\Psi y^*_{Blue-Search} = 7.330$.

Doing the same for Blue, we obtain the following.

m=0	1	2	3	4	5	6
C_i^0	4.21	6.56	4.21	6.56	6.56	6.56
E_i^0	2.37	1.62	2.37	3.05	4.21	3.05

Table 18. Example II—Blue's Expected Survival Time with No Moves

Table 19 shows the values of C, M, E, N, and V as given in Chapter III.

m=35	1	2	3	4	5	6
C_i^{35}	7.24	7.70	7.43	7.80	7.83	7.80
M_i^{35}	0	3.82	0	3.54	3.45	3.54
E_i^{35}	7.24	7.24	7.43	7.39	7.44	7.38
N_i^{35}	0	0	0	0	0	0
V_i^{35}	7.24	7.24	7.43	7.39	7.44	7.38

Table 19. Example II—Blue's Expected Survival Time with Multiple Moves

As with Blue in Example 1, all D and N are zero, since there is little risk to Red or Blue to move, if they find themselves in the same cell as their adversary. We will continue to define Blue's optimal strategy as *y* and Red's as *x*. The optimal mixed strategy for each player considering only the Ambush model thus becomes

	1	2	3	4	5	6
$x^*_{\mathrm{Re}d-Ambush}$	0	0.0426	0	0.2862	0.3872	0.2841
$y^*_{Blue-Ambush}$	0	0.2253	0	0.2545	0.2659	0.2543

Table 20. Example II—Optimal Ambush Strategies

The value of the Ambush model is $V_{Ambush} = y^{*T}_{Blue-Ambush}\Lambda x^*_{\mathrm{Re}d-Ambush} = 7.6776$ and this also gives us the time Blue can expect to survive without an ambush using the available routes. Unlike our previous example $V_{Ambush} > V_{Search}$, leading us to believe that Blue has a slight advantage in this scenario and can expect to live longer.

As before, we obtain the following bi-matrix consisting of Ψ and Λ^T; each cell contains a value from each individual payoff matrix (ψ, λ).

	1	**2**	**3**	**4**	**5**	**6**
1	(6.86, 7.24)	(6.86, 7.70)	(6.86, 7.43)	(6.86, 7.79)	(6.86, 7.83)	(6.86, 7.79)
2	(6.96, 7.24)	(6.96,7.24)	(6.96, 7.43)	(6.96, 7.79)	(6.96, 7.83)	(6.96, 7.79)
3	(7.49, 7.24)	(7.49, 7.70)	(6.89,7.43)	(7.49, 7.79)	(7.49, 7.83)	(7.49, 7.79)
4	(7.13, 7.24)	(7.13, 7.70)	(7.13, 7.43)	(7.13,7.39)	(7.13, 7.83)	(7.13, 7.79)
5	(7.55, 7.24)	(7.55, 7.70)	(7.55, 7.43)	(7.55, 7.79)	(7.00, 7.44)	(7.55, 7.79)
6	(7.52, 7.24)	(7.52, 7.70)	(7.52, 7.43)	(7.52, 7.79)	(7.52, 7.83)	(6.95,7.38)

Table 21. Example II—Bimatrix Model

The pure Nash Equilibria are highlighted in Table 21. Interestingly, both Red and Blue would prefer to use Route 5 as it provides them the greatest value if their adversary is not along that route. To avoid confrontation, which would decrease their survival time, Red and Blue would benefit from using the route pairs (5,4), (6,5), or (5,6) to maximize their survival times. As mentioned previously, a pure strategy is not viable over time as it provides the adversary a clearer picture of where to find you. We see through our non-linear programming that a mixed strategy provides Red and Blue both with greater values for their individual games though at a risk that they might take the same route at the same time.

Using non-linear optimization we will start our search for mixed Nash Equilibria with the values of the individual games $V_{Search} = 7.330$ and $V_{Ambush} = 7.6776$. This produces the following optimal strategies:

(7.330,7.678)	**1**	**2**	**3**	**4**	**5**	**6**
$x^*_{\mathrm{Re}d}$	0	0	0.8974	0	0.1026	0
y^*_{Blue}	0	0	0	0.8383	0.1091	.0526

Table 22. Example II—Optimal Bimatrix, Risk Adverse Strategies

Using these mixed strategies; Red and Blue obtain the following values for the game: $V_{\mathrm{Red}} = x^{*T}\Psi y^{*} = 7.49$ and $V_{Blue} = x^{*T}\Lambda y^{*} = 7.79$. Note how their mixed strategies give them a greater payoff than the pure strategies. This comes from an interesting choice for how they randomly choose the route they will take.

From Table 21, Red's route preference (from highest value to lowest) should be 5-6-3-4-2-1. Blue's preference should be 5-4-6-1-2-3 where 4 and 6 could be interchanged, since they have the same value. Interestingly, the mixed strategies in Table 22 show that Red will avoid 6, even though it is his second highest valued route, to avoid Blue. The values for Route 5 are great enough that both Red and Blue will risk taking route 5 approximately 10% of the time to increase their overall value leading them to increase their overall values from the pure strategies.

As in Example 1, we set our initial value at *(0,0)* to represent a more aggressive game where each player wishes to obtain a strategy that brings their opponent's values to zero. Doing so produces the following optimal strategies:

(0,0)	**1**	**2**	**3**	**4**	**5**	**6**
x^{*}_{Red}	0	0	0.8974	0	0.1026	0
y^{*}_{Blue}	0	0	0	0.8383	0.1091	.0526

Table 23. Example II—Optimal Bimatrix, Risk Prone Strategies

Clearly, there is no change in optimal strategies from our previous initial starting point and, therefore, the values of the game for Red and Blue go unchanged. In fact, by varying our initial point we can see that this mixed strategy is relatively stable. Each player can be assured of the outcome regardless of the aggressiveness of their adversary. As before, Blue can then use this information to route his convoy using y^{*} knowing which routes he is most likely to encounter Red on.

V. TOPICS FOR FURTHER RESEARCH

A. APPLICATION

Practical application of this method is reliant on determining the initial probabilities of indirect and direct detection for each route under consideration. While this may be impossible to do with any certainty, approximations based on the relative risk each route presents can provide a starting point. This model also offers the ability to determine the effect, if any; indirect detection methods (rewards, humanitarian efforts, etc.) have on the enemy's strategy for route selection in the face of trying to engage a convoy. Also, by assigning cost to both direct and indirect measures, the commander can best determine which investment returns the greatest survival times for his convoy.

B. POSSIBLE FOLLOW-ON RESEARCH

This model provides only a starting point in exploring the relationships between the indirect and direct aggressiveness each player exhibits in trying to minimize their opponent's survival time while maximizing their own. This model could easily be adopted for cities where Blue wishes to dissuade enemy activity through both direct and indirect means. In this scenario, an optimal control problem is clearly present. What is the balance of direct and indirect aggressiveness that minimizes the time Red stays in the city that also allows Blue to maximize resources? Another avenue of research is the relationship between Blue's aggressiveness ($\boldsymbol{\alpha},\boldsymbol{\beta}$) and Red's aggressiveness ($\boldsymbol{\gamma}$). This can be applied to the classic problem faced by law enforcement. As each player gets increasingly aggressive in their direct attempts to eliminate their opponent, they are greeted with increasing direct aggressiveness from their opponent. A indirect aggressive approach may be more appropriate and this model provides for exploring that option. Finally, another possible research path is to apply this model to a network of routes where flow analysis can be combined with the expected survival times along each route to determine the best overall route to take. In conclusion, this model can be readily adopted for a myriad of problems where two parties have conflicting goals and two methods of achieving those goals.

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